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On the radiation of short surface waves by a heaving circular cylinder

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SUMMARY

A long circular cylinder half immersed in the free surface of an ideal fluid undergoes small time periodic motions. The method of matched asymptotic expansions is used to give a solution in the high frequency limit. Of particular interest are the surface waves generated by this motion, and a three term asymptotic series for their amplitude is found. It is proved that there are no eigensolutions of the infinite vertical barrier problem containing waves which are purely outgoing, and it is shown how this can be used to predict the wave amplitude to a higher order than that of the matching solution.

1. Introduction

A long circular cylinder, half immersed in the free surface of a fluid of infinite extent with its generators horizontal, executes a small time periodic heaving motion. The aim of this note is to give details of a perturbation solution of this wave-maker problem in the high frequency (i.e., short wave) limit.

A rigorous treatment of this problem has been given by Ursell [1], who obtained a first order estimate of the amplitude of the generated surface waves and the virtual mass of the cylinder. Later, Hermans [2] developed a "straightforward" asymptotic method to deal with the problem, and determined a three term asymptotic series for the wave amplitude. This method was intuitively similar to Van Dyke's [3] method of matched asymptotic expansions in that the solution was approximated by different asymptotic expansions in different regions. However, the author claimed that to the order of the solution he found, it was unnecessary to make the asymptotic expansions in adjacent regions satisfy a matching principle.

Coordinates are chosen with the z-axis along the axis of the cylinder, the y-axis directed into the fluid, and the x-axis lying in the free surface. The lines $x=a$, $y=0$ and $x=-a$, $y=0$ where the cylinder meets the free surface are denoted by P and Q respectively. We take the vertical heaving velocity to be $V\cos\omega t$, and seek a two-dimensional irrotational velocity potential of the form $\mathrm{Re}\{\phi(x, y)\exp(-\mathrm{i}\omega t)\}$, which is finite at P and Q. Suppressing the time factor $\exp(-\mathrm{i}\omega t)$, the potential $\phi(x, y)$ is specified by the linearized conditions

$$\phi_{xx}+\phi_{yy}=0\,, \qquad \text{in the fluid}\,, \tag{1.1}$$

$$\phi_n = Vy/a\,, \quad \text{on } S\,, \tag{1.2}$$

$$\phi+\varepsilon\phi_y=0\,, \qquad \text{on the free surface}\,, \tag{1.3}$$

where subscripts denote partial derivatives, n is the outward normal from the submerged part S of the cylinder's surface, and $\varepsilon=g/\omega^2$ is $(1/2\pi)$ of the wavelength of the generated surface waves. The conditions

$$\delta(\partial\phi/\partial\delta)\to 0 \text{ as } \delta\to 0\,, \text{ where } \delta^2=(x\pm a)^2+y^2\,, \tag{1.4}$$

$$\phi\sim A_\pm \exp\{(\pm\mathrm{i}x-y)/\varepsilon\} \text{ as } x\to\pm\infty\,, \tag{1.5}$$

ensure boundedness at P and Q, outgoing waves at infinity, and hence the uniqueness of the potential ϕ.

The method used here is the systematic method of matched asymptotic expansions developed by Van Dyke [3] as applied to short surface wave problems by Leppington [4, 5, 6], Ayad [7] and Alker [8]. The basic idea is that the fluid region can be covered by a number of overlapping domains, in each of which an asymptotic approximation for the potential $\phi(x, y)$ may be found, the argument running briefly as follows.

In the "outer region", points at distances $\gg \varepsilon$ from the free surface, the potential ϕ is written as an asymptotic expansion involving powers and logarithms of ε. This expansion is formally substituted into Eqn. (1.3) to obtain subsidiary boundary conditions of the form

$$\phi_j(x, 0) = f(x), \qquad |x| > a, \tag{1.7}$$

where $f(x)$ is either zero or the derivative of a previous term. Although this outer expansion is expected to be valid throughout most of the fluid, it is clear that the potentials ϕ_j do not contain surface waves and the expansion fails near the surface. In particular, eigensolutions with singularities at P and Q may freely be added to each term of the expansion, their coefficients to be determined by matching. The limit of the outer expansion approaching these points is of the utmost importance however, and use is made of the polar coordinates (δ, θ) defined at P by

$$(x-a, y) = (\delta \cos \theta, \delta \sin \theta). \tag{1.8}$$

In the vicinity of the two intersection points P and Q, the potential will be sensitive to the wave bearing nature of the free surface, but will depend primarily on the local geometry of S near these points. This suggests that the potential in these "inner regions", points at distances $\ll a$ from P and Q, is expected to vary on a wave-length scale, and for detailed examination we take coordinates

$$(X, Y) = (R \cos \theta, R \sin \theta) = ((x-a)/\varepsilon, y/\varepsilon), \tag{1.9}$$

and the inner potential

$$\Phi(X, Y) = \phi(x, y) \tag{1.10}$$

is defined. Note that due to symmetry we need only consider one of the inner regions. In the inner region near P it is convenient to replace the boundary condition on S by a new condition on $X=0$, by expanding S and Φ in Taylor series in X. The actual details of this boundary condition and of the subsequent asymptotic expansion posed for the inner potential Φ are left to the main text, although it is noted here that each term of the inner expansion takes the form of a classical wave maker problem, the normal derivative on the vertical wave maker being given as a function of previously determined terms. Although the potential $\Phi(X, Y)$ must be bounded at the origin, there is no such condition as R increases, so eigensolutions (unbounded at infinity) must be added to each "wave maker" solution. The coefficients of the eigensolutions in the inner and outer expansion are determined when the expansions are matched together.

The matching principle to be used is a modified version of that proposed by Van Dyke [3]: the modification due to Crighton and Leppington [9] stipulates that all terms of the form $\varepsilon^\alpha \log \varepsilon$, $\varepsilon^\alpha \log \log \varepsilon$ must be grouped with ε^α for matching purposes. We first define the expansion of the inner potential $\Phi(R, \theta; \varepsilon)$ up to and including terms of order ε^s by $\Phi^{(s)}(R, \theta; \varepsilon)$. Then in order to match the inner potential $\Phi(R, \theta; \varepsilon)$ with the outer potential $\phi(\delta, \theta; \varepsilon)$, we take the limit of $\Phi^{(s)}$ as $R \to \infty$, and replace R by δ/ε. This is expanded in ε (for fixed δ), and truncated to include terms of order up to and including ε^t, and the resulting series is denoted by $\Phi^{(s,t)}$. Similarly by replacing δ by εR in $\phi^{(t)}$, expanding and truncating after ε^s, we obtain $\phi^{(t,s)}$. The matching condition is

$$\Phi^{(s,t)} \equiv \phi^{(t,s)}, \tag{1.11}$$

for any s, t.

Finally, the outer expansion is extended up to the free surface (for points $\gg \varepsilon$ from P and Q) by simply continuing the surface waves—initially valid only in the inner regions—over the whole free surface.

It is worthwhile to consider the values s and t must take to determine all of the coefficients of the eigensolutions in the two expansions. For to ensure the matching condition (1.11) determines all inner eigensolution coefficients, t must be large enough to make $(\varepsilon^r E)^{(s,t)}$ non zero, where $r \leqq s$ and E is an inner eigensolution. The critical case is obviously $r=s$ with the lowest order eigensolution ($E=R\sin\theta-1$): this demands $t \geqq s-1$. Similarly eigensolutions of the outer region require $s \geqq t-1$. As our main interest is in the surface waves and hence the inner solutions, we choose $s=t+1$.

If, however, only the amplitude of the surface waves is required and not a full asymptotic solution, we can use a simple but useful result concerning the eigensolutions of the inner region. It is proved in the appendix that there are no eigensolutions containing waves which are purely outgoing. So if a term in the inner expansion has an outgoing wave, its amplitude is determined by the wave maker solution, which in turn is only dependent on previous terms of the inner expansion—in fact of scaling ε lower (see boundary condition (2.8)). Thus to determine the generated waves to order ε^N, we need only find the inner expansion to ε^{N-1} and the outer to ε^{N-2}. This result readily applies to other surface wave problems where a body intersects the free surface normally: if the geometry of the body near the intersection point $(a, 0)$ has the form

$$x-a = \sum_{r=M}^{\infty} \frac{\alpha_r}{r!} y^r$$

for $2 \leqq M < \infty$, then the wave amplitude may be calculated to an order ε^{M-1} higher than that at which the inner solution has been matched. The case of plane vertical intersections ($M=\infty$) cannot be treated in this way, though a separate treatment (using matched asymptotic expansions) has been given by Ayad [7].

In order to demonstrate the self-consistency of the system however, a full matching solution will be given with the inner expansion found to ε^3 and the outer to ε^2. The calculated asymptotic series for the wave amplitude differs from that of Hermans [2] in both magnitude and phase, but far from suggesting any unreliability in singular perturbation type methods, this disagreement emphasizes the need for a thorough and systematic approach. In comparison with Hermans' work, note should be taken of the omission of a factor of π from Eqn. (4.3) onwards and that our outer potential ϕ_1 suggests that the y coefficient of Ψ_1 should be $-4U/\alpha\pi$ and not $-5U/a\pi$.

2. Calculation of the asymptotic expansions

As Leppington [5] has already given a formulation of this problem to first order, details will be kept to a minimum.

In the outer expansion, we certainly expect terms with unit and ε scaling (from conditions (1.2) and (1.3)), hence we pose

$$\phi \sim \phi^{(1)} = \phi_0 + \varepsilon\phi_1 + h^{(1)}(\varepsilon)\phi_e\,, \tag{2.1}$$

where the term $h^{(1)}(\varepsilon)\phi_e$ is added to indicate the possibility of terms of different scalings to order ε. Clearly the potentials ϕ_e will be eigensolutions of the outer problem. Substitution of Eqn. (2.1) into the conditions (1.1)–(1.3) yields the following conditions for the harmonic ϕ_i:

$$\begin{aligned} &\phi_{0n} = Vy/a\,, \quad \phi_{1n} = 0\,, \qquad \text{on } S\,,\\ &\phi_0 = 0\,, \qquad \phi_1 = -\phi_{0y}\,, \quad y=0\,,\ |x|>a\,. \end{aligned} \tag{2.2}$$

We find

$$\phi_0 = -a^2 Vy/(x^2+y^2) + \text{Eigensolutions}\,.$$

Now the symmetric eigensolutions of the outer region are:

$$E_n = \mathrm{Im}\left[\left(\frac{z-a}{z+a}\right)^{2n+1} - \left(\frac{z+a}{z-a}\right)^{2n+1}\right] \sim \left(\frac{2a}{\delta}\right)^{2n+1} \sin(2n+1)\theta\,,\ \text{as } \delta\to 0\,, \tag{2.3}$$

where $z=x+iy$, $n \geqq 0$ is an integer. However, as the existence of these eigensolutions in ϕ_0

would require terms of order ε^{-2n-1} in the inner expansion we conclude

$$\phi_0 = -a^2 Vy/(x^2+y^2)\,. \tag{2.4}$$

Using Eqns. (2.2) and the Green's function

$$L(s,t;x,y) = \tfrac{1}{2}\log\frac{(s-x)^2+(t+y)^2}{(s-x)^2+(t-y)^2} + \tfrac{1}{2}\log\frac{(a^2-sx+ty)^2+(tx+sy)^2}{(a^2-sx-ty)^2+(tx-sy)^2}\,, \tag{2.5}$$

we find

$$\begin{aligned}\phi_1(x,y) = \frac{Va^2}{\pi}\Bigg[&\frac{x^2-y^2}{(x^2+y^2)^2}\left(\pi+\tan^{-1}\frac{x-a}{y} - \tan^{-1}\frac{x+a}{y}\right)+\\ &+\frac{x^2-y^2}{a^4}\left(\pi-\tan^{-1}\frac{x^2+y^2-ax}{ay} - \tan^{-1}\frac{x^2+y^2+ax}{ay}\right)+\\ &+xy\left(\frac{1}{(x^2+y^2)^2}+\frac{1}{a^4}\right)\log\frac{(x-a)^2+y^2}{(x+a)^2+y^2}+\frac{2y}{a}\left(\frac{1}{(x^2+y^2)}+\frac{1}{a^2}\right)\Bigg]\\ &+\text{Eigensolutions}\,.\end{aligned} \tag{2.6}$$

In particular, as $\delta\to 0$,

$$\begin{aligned}\phi^{(1)} \sim V\Bigg[&-\delta\sin\theta+\frac{\delta^2}{a}\sin 2\theta-\frac{\delta^3}{a^2}\sin 3\theta+\dots\\ &+\varepsilon\left(1+\frac{4\delta}{a\pi}\left(\log\frac{\delta}{2a}\sin\theta+\left(\theta-\frac{\pi}{2}\right)\cos\theta\right)+\frac{4\delta}{a\pi}\sin\theta\right.\\ &\left.-\frac{2\delta^2}{a^2\pi}\left(\log\frac{\delta}{2a}\sin 2\theta+\theta\cos 2\theta\right)+\frac{\delta^2}{a^2\pi}(3\pi\cos 2\theta-4\sin 2\theta)+\dots\right)\Bigg]\\ &+\varepsilon E_n(\delta,\theta)\,.\end{aligned} \tag{2.7}$$

We now construct the new wall boundary condition in the inner region. Using the inner potential Φ and the co-ordinates (X, Y) as defined by Eqns. (1.9), (1.10), near P the condition (1.2) becomes

$$\Phi_X + F'(Y)\Phi_Y = \varepsilon VF'(Y) \quad \text{on} \quad X+F(Y)=0\,,$$

where $-X=F(Y)=\varepsilon Y^2/2a+\varepsilon^3 Y^4/8a^3+\dots$ is the Taylor expansion of S near P. Expanding $\Phi(X, Y)$ as a Taylor series in X about $X=0$, and replacing X by $-F(Y)$ we find

$$\begin{aligned}&\Phi_X+\frac{\varepsilon}{2a}(2Y\Phi_Y-Y^2\Phi_{XX})+\frac{\varepsilon^2}{8a^2}(Y^4\Phi_{XXX}-4Y^3\Phi_{YX})+\\ &+\frac{\varepsilon^3}{8a^3}\left(12Y^3\Phi_Y-Y^4\Phi_{XX}-\frac{Y^6}{6}\Phi_{XXXX}+Y^5\Phi_{YXX}\right)=\frac{\varepsilon^2}{a}VY+\frac{\varepsilon^4}{2a^3}VY^3+\dots,\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{on } X=0\,.\end{aligned} \tag{2.8}$$

Replacing δ by εR in Eqn. (2.7) and expanding in ε, we find terms with scalings ε, $\varepsilon^2\log\varepsilon$, ε^2, ... which suggest the following expansion for the inner potential Φ,

$$\Phi \sim \Phi^{(2)} = \varepsilon\Phi_0+\varepsilon^2\log\varepsilon\,\Phi_1+\varepsilon^2\Phi_2+l^{(2)}(\varepsilon)\Phi_e\,. \tag{2.9}$$

On substituting this expansion into the conditions (1.1), (1.3) and (2.8), we find that the potentials $\Phi_i(X, Y)$ are harmonic and must satisfy the conditions

$$\Phi_i+\Phi_{iY}=0\,, \qquad Y=0\,,\ X>0\,, \tag{2.10}$$

$$\Phi_{0X}=\Phi_{1X}=0\,, \qquad X=0\,,\ Y>0\,, \tag{2.11}$$

$$\Phi_{2X}=-(1/2a)(2Y\Phi_{0Y}-Y^2\Phi_{0XX})+VY/a\,, \qquad X=0\,,\ Y>0\,. \tag{2.12}$$

It is now clear that each inner potential is the solution of a classical wave maker problem, the solutions given in terms of the Green's function G are

$$\Phi_i(X, Y) = \int_0^\infty \Phi_{iX'}(0, Y')G(0, Y'; X, Y)dY', \tag{2.13}$$

where

$$G(X', Y'; X, Y) = G_0(X', Y'; X, Y) + G_0(X', Y'; -X, Y) \tag{2.14}$$

is defined by

$$\begin{aligned} G_0(X', Y'; X, Y) = &-\mathrm{i}\exp\{\mathrm{i}|X'-X| - (Y'+Y)\} + \\ &+ \frac{1}{4\pi}\log\left\{\frac{(X'-X)^2+(Y'+Y)^2}{(X'-X)^2+(Y'-Y)^2}\right\} \\ &- \frac{1}{\pi}\int_0^\infty \frac{t\cos(Y'+Y)t - \sin(Y'+Y)t}{1+t^2}\,\mathrm{e}^{-|X'-X|t}dt\,. \end{aligned}$$

We find

$$\Phi_0 = A(R\sin\theta - 1), \tag{2.15}$$

$$\Phi_1 = B(R\sin\theta - 1), \tag{2.16}$$

$$\Phi_2 = -(2A/a)(\Psi + C(R\sin\theta - 1)), \tag{2.17}$$

where

$$\begin{aligned} \Psi = &\tfrac{1}{2}R^2\sin 2\theta - R\cos\theta + \frac{2}{\pi}R(\sin\theta\log R + \theta\cos\theta) \\ &-\frac{2}{\pi}\log R - \frac{2}{\pi} - 2\mathrm{i}\,\mathrm{e}^{iX-Y} - \frac{2}{\pi}\int_0^\infty \frac{t\cos Yt - \sin Yt}{1+t^2}\,\mathrm{e}^{-Xt}dt\,. \end{aligned} \tag{2.18}$$

As the integral in Eqn. (2.13) only converges if $\Phi_{iX'} \to 0$ as $Y' \to \infty$, simple harmonic potentials satisfying the free surface condition (2.10) are subtracted from the potential Φ_2 before Eqn. (2.13) is used. In this case, allowing for logarithmic singularities at the origin, the condition (2.12) can be satisfied exactly. The singularity is then removed by the addition of a suitable multiple of $G(0, 0; X, Y)$, and hence Ψ is obtained. Note that higher eigensolutions of the form $R^{2n+1}\sin(2n+1)\theta - (2n+1)R^{2n}\cos 2n\theta$ should have been added to each potential (especially before the use of Eqn. (2.12)) but as each would require a term in the outer expansion of order ε^{-1} or worse, they have been omitted.

The inner and outer expansions are now matched together using the condition $\Phi^{(2,1)} \equiv \phi^{(1,2)}$. We find

$$h^{(1)}(\varepsilon) = l^{(2)}(\varepsilon) = 0\,, \; A = -V\,, \; B = 4V/a\pi\,, \; C = \frac{2}{\pi}(1 - \log 2a)\,, \tag{2.19}$$

and the outgoing waves are

$$\phi \sim -(4V/a\pi)\varepsilon^2\,\mathrm{e}^{-\mathrm{i}a/\varepsilon}\,\mathrm{e}^{(\pm\mathrm{i}x - y)/\varepsilon}\,, \text{ as } x \to \pm\infty\,. \tag{2.20}$$

To continue, we pose for the outer expansion:

$$\phi \sim \phi^{(2)} = \phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + h^{(2)}(\varepsilon)\phi_e\,. \tag{2.21}$$

The potential ϕ_2 is harmonic and must satisfy

$$\phi_{2n} = 0 \text{ on } S\,, \; \phi_2 = -\phi_{1y} \qquad y = 0\,, \; |x| > a\,. \tag{2.22}$$

Using the Green's function, Eqn. (2.3) we obtain

$$\begin{aligned} \phi_2 = -\frac{4V}{a\pi} - \frac{4}{a\pi}\phi_1 + \frac{Va^2}{\pi^2}\Big[&\pi^2 y(3x^2 - y^2)\left(\frac{1}{(x^2+y^2)^3} + \frac{1}{a^6}\right) + \frac{4y}{a^2}\left(\frac{1}{(x^2+y^2)} + \frac{1}{a^2}\right) + \\ &+ T(z) - T(\bar{z}) - T(a^2/z) + T(a^2/\bar{z})\Big] + \text{Eigensolutions}\,, \end{aligned} \tag{2.23}$$

where

$$T(z) = -\frac{i}{2}\left(\frac{1}{z^3} + \frac{z}{a^4}\right)\left(\left(\log\left(\frac{z-a}{z+a}\right)\right)^2 - 2\pi i \log\left(\frac{z-a}{z+a}\right)\right). \tag{2.24}$$

Here, $z=x+iy$, the cuts in $\log(z\mp a)$ are taken from $\pm a$ to $\pm\infty$ along the real axis, and $\log(x\mp a+iy)\to\ln(x\mp a)$ as $y\to 0_+$, $x>a$. Note ϕ_2 is real. In particular,

$$\phi_2 \sim \frac{4V}{a^2\pi^2}\left\{2a\pi + a\pi\log\frac{\delta}{2a} + \delta\left[4\left(\log\frac{\delta}{2a}\sin\theta + \left(\theta-\frac{\pi}{2}\right)\cos\theta\right) - \frac{\pi}{2}\cos\theta + \right.\right.$$
$$+ 2\log\frac{\delta}{2a}\,\theta\cos\theta + \left(\log^2\frac{\delta}{2a} - \theta^2\right)\sin\theta - \pi\left(\log\frac{\delta}{2a}\cos\theta - \theta\sin\theta\right)$$
$$\left.\left. - (\pi^2-2)\sin\theta\right] + \ldots\right\} + E_n(\delta,\theta), \text{ as } \delta\to 0. \tag{2.25}$$

The presence of the $(\log(\delta/2a))^2$ term implies the existence of a potential with scaling $\varepsilon^3\log^2\varepsilon$ in the inner expansion, and so we pose

$$\Phi \sim \Phi^{(3)} = \varepsilon\Phi_0 + \varepsilon^2\log\varepsilon\Phi_1 + \varepsilon^2\Phi_2 + \varepsilon^3\log^2\varepsilon\Phi_3 + \varepsilon^3\log\varepsilon\Phi_4 + \varepsilon^3\Phi_5 + l^{(3)}(\varepsilon)\Phi_e. \tag{2.26}$$

As before the boundary conditions for the new potentials are obtained by substituting the above expansion into Eqn. (2.8):

$$\Phi_{3X} = \Phi_{eX} = 0, \tag{2.27}$$
$$\Phi_{4X} = -(1/2a)(2Y\Phi_{1Y} - Y^2\Phi_{1XX}), \tag{2.28}$$
$$\Phi_{5X} = -(1/2a)(2Y\Phi_{2Y} - Y^2\Phi_{2XX}) - (1/8a^2)(Y^4\Phi_{0XXX} - 4Y^3\Phi_{0YX}), \text{ on } X=0,\ Y>0. \tag{2.29}$$

(The wave amplitude to ε^3 may now be found directly: the potentials Φ_3 and Φ_e are clearly wave free, Φ_4 has wave amplitude exactly $-4/a\pi$ times that of Φ_2, and Φ_5 has outgoing wave $-2i\,e^{iX-Y}\int_0^\infty \Phi_{5X'}e^{-Y'}dY'$). It follows that

$$\Phi_3 = D(R\sin\theta - 1), \tag{2.30}$$
$$\Phi_4 = -(4V/a^2\pi)(\Psi + F(R\sin\theta - 1)). \tag{2.31}$$

Substitution of Eqns. (2.15) and (2.17) into Eqn. (2.29) gives for the potential Φ_5,

$$\Phi_{5X}(0,Y) = -\frac{2V}{a^2\pi}(5-2\log 2a)Y - \frac{4V}{a^2\pi}Y\log Y + \frac{2V}{a^2\pi}$$
$$-\frac{2iV}{a^2}(2Y-Y^2)e^{-Y} - \frac{2V}{a^2\pi}\frac{\partial}{\partial Y}\left(Y^2\int_0^\infty \frac{t^2\sin Yt + t\cos Yt}{1+t^2}dt\right). \tag{2.32}$$

Suitable harmonic potentials are subtracted from the potential Φ_5, then use of the Green's function Eqn. (2.14) gives:

$$\Phi_5(X,Y) = \frac{2V}{a^2\pi}\left\{(2\log 2a - 5)\Psi - 2\left(H + \int_0^\infty G(0,Y';X,Y)(Y'\log Y' - H_{X'}(0,Y'))dY'\right) + \right.$$
$$+ 2\left(R(\log R\sin\theta + \theta\cos\theta) - \log R - 1 - i\pi e^{iX-Y} - \int_0^\infty \frac{t\cos Yt - \sin Yt}{1+t^2}e^{-Xt}dt\right)$$
$$- i\pi\int_0^\infty G(2Y'-Y'^2)e^{-Y'}dY' + \int_0^\infty G_{Y'}Y'^2\int_0^\infty \frac{t^2\sin Y't + t\cos Y't}{1+t^2}dt\,dY' +$$
$$\left. + \mu(R\sin\theta - 1) + \nu(R^3\sin 3\theta - 3R^2\cos 2\theta)\right\}. \tag{2.33}$$

Where H is given in terms of $Z = X + iY$ by

$$H(X, Y)=\mathrm{Re}\left\{-\frac{\mathrm{i}}{2}Z^2\log(Z+1)+\frac{\mathrm{i}}{4}Z^2-\frac{\mathrm{i}}{\pi}Z(\log(-\mathrm{i}(Z+1)))^2\right.$$
$$-\frac{1}{\pi}(\log(Z+1))^2-\frac{2}{\pi}\frac{Z}{Z+1}\log(Z+1)+\frac{2\mathrm{i}}{\pi}\log(Z+1)+$$
$$\left.+\frac{4+\pi Z}{2\pi(Z+1)}+\frac{\pi}{4}+\frac{1}{2}+\frac{\mathrm{i}Z}{2}\right\}. \tag{2.34}$$

In particular, as $R\to\infty$,

$$\Phi_5\sim\frac{2V}{a^2\pi}\left\{((14-4(\log 2a+\gamma))\mathrm{i}-\tfrac{1}{2}\pi)\mathrm{e}^{\mathrm{i}X-Y}-R^2(\log R\sin 2\theta+\theta\cos 2\theta)+\right.$$
$$+(\log 2a-2)R^2\sin 2\theta-\frac{2}{\pi}R\left[\left(\log^2 R-\left(\theta-\frac{\pi}{2}\right)^2\right)\sin\theta+2\left(\theta-\frac{\pi}{2}\right)\cos\theta\log R\right]+$$
$$+\frac{4}{\pi}(\log 2a-2)R(\log R\sin\theta+\theta\cos\theta)+(5-2\log 2a)R\cos\theta+\frac{2}{\pi}(\log^2 R-\theta^2)+$$
$$+\frac{4}{\pi}(3-\log 2a)\log R+\frac{4}{\pi}(2-\log 2a)-\frac{\pi}{2}+\mu(R\sin\theta-1)+$$
$$\left.+\nu(R^3\sin 3\theta-3R^2\cos 2\theta)+o(1)\right\}. \tag{2.35}$$

The eigensolution coefficients D, F, μ and ν are determined using the matching condition $\Phi^{(3,2)}\equiv\phi^{(2,3)}$ to be

$$D=-4V/a^2\pi^2,\quad F=2(2-\log 2a)/\pi,$$
$$\mu=2(3\pi^2/4-2+4\log 2a-\log^2 2a)/\pi,\quad \nu=-\pi/2. \tag{2.36}$$

Also, $h^{(2)}(\varepsilon)=l^{(3)}(\varepsilon)=0$.

The solution is satisfactorily completed, and the outgoing wave trains are given by

$$\phi\sim\frac{2V}{a^2\pi}\left\{\begin{array}{l}-2a\pi\mathrm{i}\varepsilon^2+4\mathrm{i}\varepsilon^3\log\varepsilon+\\+((14-4(\log 2a+\gamma))\mathrm{i}-\tfrac{1}{2}\pi)\varepsilon^3\end{array}\right\}\mathrm{e}^{-\mathrm{i}a/\varepsilon}\mathrm{e}^{(\pm\mathrm{i}x-y)/\varepsilon},\quad\text{as } x\to\pm\infty. \tag{2.37}$$

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Appendix: Eigensolutions of the infinite vertical barrier problem

We define an eigensolution of a sloping beach problem to be a solution satisfying homogeneous boundary conditions with no incoming wave, which is harmonic in the fluid and bounded at the intersection of the free surface and the beach. As we are only interested in eigensolutions which will match with some outer potential we may assume that the solution minus its outgoing wave is bounded by R^α, for large R. It is well known that if $\alpha<0$ the eigensolution is identically zero, but if α is merely restricted to be a positive integer, eigensolutions do exist. With all sloping beach problems except the vertical case, these solutions have outgoing waves. We now prove that eigensolutions of a vertical beach problem are wave-free.

We seek potentials Φ, harmonic in the quarter plane $X>0$, $Y>0$, which satisfy

$$\Phi+\Phi_Y=0,\qquad Y=0, X>0,$$
$$\Phi_X=0,\qquad X=0, Y>0,$$
$$R(\partial\Phi/\partial R)\to 0,\ \text{as } R\to 0,$$

$$\Phi \text{ has no incoming waves}, \tag{A.1}$$

$$\Phi = O(R^N) \text{ as } R \to \infty,\ 0 \leqq \theta \leqq \pi/2 \text{ for some integer } N. \tag{A.2}$$

To distinguish between the wave-like and wave-free parts of Φ (for large R), we note that the surface wave potentials e^{-iX-Y} and e^{iX-Y} (corresponding to incoming and outgoing waves respectively) may be differentiated any number of times and still remain $O(1)$ as $R \to \infty$, $\theta = 0$. Thus, due to (A.1) and (A.2), there exists an integer M and a complex number A (the amplitude of the outgoing wave) such that

$$\frac{\partial^{2M}}{\partial R^{2M}}(\Phi - A\,e^{iX-Y}) = O(1/R), \text{ as } R \to \infty,\ 0 \leqq \theta \leqq \pi/2. \tag{A.3}$$

Recall that if the potential Ω is an eigensolution which satisfies

$$\Omega - A\,e^{iX-Y} = O(1/R), \text{ as } R \to \infty,\ 0 \leqq \theta \leqq \pi/2, \tag{A.4}$$

then Ω is identically zero, and $A = 0$.

We now prove that the existence of the eigensolution Φ with non zero constant $A = A^*$ in (A.3), implies the existence of the eigensolution Ω with outgoing wave $A^* e^{iX-Y}$ which satisfies (A.4).

Firstly, the potential Φ is harmonic in $X \geqq 0$, $Y \geqq 0$. The Schwarz Reflection Principle can be used to make an analytic continuation of Φ across $X = 0$, $Y > 0$ and $Y = 0$, $X > 0$, so we need only show that Φ is harmonic at the origin. Using Green's Theorem on Φ and G (the exact Green's function (2.8)) in the quarter circle bounded by

$$\Gamma_1: 0 \leqq X \leqq 1,\ Y = 0\ ; \quad \Gamma_2: 0 \leqq Y \leqq 1, X = 0\ ; \quad \Gamma: 0 \leqq \theta \leqq \pi/2,\ R = 1\ ;$$

we find

$$\Phi(X, Y) = \int_\Gamma (G(X', Y'; X, Y)\Phi_{R'}(X', Y') - \Phi(X', Y')G_{R'}(X', Y'; X, Y))\,ds. \tag{A.5}$$

Now as Φ and Φ_R are harmonic on Γ, and $G(\cos\theta', \sin\theta'; X, Y)$ is harmonic for $X^2 + Y^2 < \frac{1}{4}$ say, (A.5) provides an analytical continutation of Φ into this circle about the origin, and in particular Φ is harmonic at the origin. So Φ and any derivative of Φ is harmonic in $X \geqq 0$, $Y \geqq 0$.

Finally, if Φ is an eigensolution with wave amplitude A^*, then Φ_{YY} is an eigensolution with the same wave amplitude. For

$$\Phi_{YYX}(0, Y) = \frac{\partial^2}{\partial Y^2}\Phi_X(0, Y) = 0,$$

$$\Phi_{YY}(X, 0) + \Phi_{YYY}(X, 0) = -\frac{\partial^2}{\partial X^2}(\Phi(X, 0) + \Phi_Y(X, 0)) = 0.$$

Hence $\Omega = (\partial^{2M}/\partial Y^{2M})\Phi$ is also an eigensolution, but

$$\Omega - A^* e^{iX-Y} = O(1/R), \text{ as } R \to \infty \qquad 0 \leqq \theta \leqq \pi/2,$$

where $A^* \neq 0$. Contradiction.

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